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Some Theorems in Topological Analyses of Electrical Networks

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Abstract

It is of importance to investigate the number of trees or cotrees and to know relative signs between trees or cotees in the theery of graphs. This paper presents some theorems which closely relate to the above two problems. They are the theorems on the decision of relative sign and those on the methods of enumeration of the number of several kinds of trees and cotrees. The applications of the result to the topogical analysis of electrical networks are stated with the examples.

Notation

It is assumed that a graph contains m+1 vertices, n branches, and s independent circuits. The graph is directed or non-directed according to cases. The following notions are used.

branches	1, 2, …, n or κ, λ, ν
independent vertices	1, 2, …, m or a, b, v
independent circuits	1, 2, …, s or m+1, m+2, …, n or p, q
incidence matrix	D^{a}_{κ}
circuit matrix	R^{κ}_{p}
unit matrix	A, A^{a}_{b}, $A^{\kappa\lambda}$, $A_{\kappa\lambda}$

D' transpose matrix of D

(κ) removing branch κ by shortenig

($\bar{\kappa}$) removing branch κ by opening

($\kappa\bar{\lambda}$) removing branch κ by shortening and branch λ by opening

e_{λ} branch voltage source S^{κ} equivalent branch current source

u_{λ} branch voltage drop i^{κ} branch current

V_a	node potential	J^p	circuit current
$z_{\kappa\lambda}$	branch impedance	$y^{\kappa\lambda}$	branch admittance

1. Introduction

If the graph contains m independent vertices, tree (or cotree) determinant is $m \times m$ (or $s \times s$) minor of incidence (or circuit) matrix whose columns correspond with a tree (or cotree), and has the invariance.

A relative sign is defined by the sign of the product of any two tree (or cotree) determinants. Several methods to decide relative sign was given up to the present. The one presented by S.Okada and R.Onodera (1) is found expedient if relative sign is decided graphically. We will give a verification to this method and adopt it as a theorem. Relative sign plays an important role when the number of trees (or cotrees) is obtained. S. Okada (1), W. T. Tutte (2) and H. M. Trent (3) indicated that the number of all trees or cotrees is given by $\det(D^a_\kappa A^{\kappa\lambda} D^b_\lambda)$ or $\det(R^\kappa_p A_{\kappa\lambda} R^\lambda_q)$. Thereafter R. Onodera (4) proved that $\det\begin{pmatrix} D^a_\kappa \\ R^\kappa_p \end{pmatrix}$ gave the number of all trees or cotrees.

We will present some theorems on the enumeration of the number of severl kinds of trees or cotrees. The results obtained are valid for the applications to the topological analysis of electrical networks. For the diagosis of nerworks, topoolgical formulas for nerwork functions are found convenient. Generally, by the use of well-known Kirchhoff's law, the action of electrical nerwork is fully described as a system of simultaneous linear equations which are formulated in terms of two variables, current and voltage. Two network functions (variables) are obtained by applying Crammer's formula in solving simultaneous equations. In practical calculations, however, it is seriously difficult not only to solve these but even to know the number of the independent equations if network is complicated.

On the other hand, in the topological analysis of a network, it has been clarified that the network functions closely depend on the connection of the network, and that the number of independent equations is decided by the number of independent vertices and circuits of the graph assoiated with a givne network diagram. In detail, the voltage functions depend on trees and current ones do on cotrees of the graph. Therefore it becomes necessary to know relative sign and the number of several kinds of trees and cotrees when the denominator and numerator of network functions are obtained.

2. Preliminaries

"Alternation and summation convention" are the operations commonly used in tensor analysis in differential geometry. The above two operations include the formal treatment of the theory of determinant. By means of alternation and summation convention, some theorems of determinant are expressed in a very available form for the applications. A brief explanation is given in what follows.

Alternation: the process of alternation over n upper or lower indices is the addition of such isomers that are obtained by all the permutations of indices and division by n!, but isomers that are obtained by odd permutations will have a negative sign. This process is denoted by symbols $[\ ]$, $\ulcorner\ \urcorner$, $\llcorner\ \lrcorner$, $\ulcorner\ \lrcorner$. If the indices have to be singled out, sign $|\ \ |$ is used in alternation. If there is a repetition in two processes of alternation such as

$$P^{[1}\cdots P_{[k}^{k}\cdots P_{r}^{r]}\cdots P_{s]}^{s},$$

$(r-k+1)!$ duplications exist with respect to each term. By multiplying each term by a factor

$$\frac{r!\,(s-k+1)!}{(r-k+1)!},$$

duplications will disappear.

Summation convention: any term in which the same index appears twice (once upper and once lower) means the sum of all such terms obtained by giving this index its complete range of values, and summation sign Σ is dropped.

Determinant: the determinant of square matrix P of order n over a field can be defined by alternation as

$$\det(P) = n!\, P_{1}^{[1} P_{2}^{2} \cdots P_{n}^{n]}$$

Binet-Cauchy's expansion: for the sake of brevity, the expansion of three matrices product will be treated.

Let P, Q, R be the matrices of order $s\times n$, $n\times n$, $n\times s$ respectively. Then

$$\begin{aligned}(\det PQR) &= s!(P_{\kappa_1}^{[1} Q_{\lambda_1}^{|\kappa_1|} R_{[1}^{|\lambda_1|})(P_{|\kappa_2|}^{2} Q_{|\lambda_2|}^{|\kappa_2|} R_{2}^{|\lambda_2|})\cdots(P_{|\kappa_s|}^{]} Q_{|\lambda_s|}^{\kappa_s} R_{s]}^{\lambda_s}) \\ &= s!P_{\kappa_1\kappa_2}^{[12}\cdots P_{\kappa_s}^{s]} Q_{\lambda_1}^{\kappa_1} Q_{\lambda_2}^{\kappa_2}\cdots Q_{\lambda_s}^{\kappa_s} R_{[1}^{\lambda_1} R_{2}^{\lambda_2}\cdots R_{s]}^{\lambda_s} \qquad (1)\end{aligned}$$

where indices κ_1, $\kappa_2 \cdots \kappa_s$, λ_1, $\lambda_2 \cdots \lambda_s$ are the integers from 1 to n.

Generalized co-factor: delete i_1, i_2, $\cdots$ i_s rows $(i_1 < i_2 < \cdots < i_s)$ and h_1,

h_2, $\cdots h_s$ columns ($h_1 < h_2 < \cdots < h_s$) from det (P) and prefix the sign $(-1)^{\alpha}$ to the result, where $\alpha = i_1 + \cdots + i_s + h_1 + \cdots h_s$. The generalized cofactor is defined by

$$C\begin{bmatrix} h_1 h_2 \cdots h_s \\ i_1 i_2 \cdots i_s \end{bmatrix} = \frac{n!\,n!}{(n-s)}!A^{[12\cdots s}_{i_1 i_2 \cdots i_s} A^{|h_1 h_2 \cdots h_s|}_{[12\cdots s} P^{s+1}_{s+1} \cdots P^{n]}_{n]} \tag{2}$$

where P is $n\times n$ matrix, and $A^{12\cdots s}_{i_1 i_2 \cdots i_s}$ means $A^{1}_{i_1} A^{2}_{i_2} \cdots A^{s}_{i_s}$ .If $s=1$, (2) comes to be ordinary co-factor of (i,h) element of det (P) written as

$$C^{h}_{i} = n!n\ A^{[1}_{i} A^{|h|}_{[1} P^{2}_{2} \cdots P \cdots ^{n]}_{n]} \tag{3}$$

Laplace's expansion: in (2) let indices i_1, i_2, $\cdots$ i_s be fixed to the first s rows of det (P), then Laplace's expansion by first s rows are represented as

$$\det (P) = s!n!\ P^{[12\cdots s]}_{h_1 h_2 \cdots h_s} A^{h_1 h_2 \cdots h_s}_{[12\cdots s} P^{[s+1}_{s+1} \cdots P^{n]}_{n]} \tag{4}$$

where indices h_1, $h_2 \cdots h_s$ range over the integers 1, 2,$\cdots$,n.

3. Theorem on the Decision of Relative Sign

The invariance of incidence or circuit matrix is given by

$$m!D^{[1}_{\kappa_1} D^{2}_{\kappa_2} \cdots D^{m]}_{\kappa_m} = \begin{cases} 1, & \text{if } \kappa_1\kappa_2\cdots\kappa_m \text{ is a tree} \\ 0, & \text{if } \kappa_1\kappa_2\cdots\kappa_m \text{ is not a tree} \end{cases} \tag{5}$$

$$s!R^{\kappa_1}_{[1} R^{\kappa_2}_{2} \cdots R^{\kappa_s}_{s]} = \begin{cases} 1, & \text{if } \kappa_1\kappa_2\cdots\kappa_s \text{ is a cotree} \\ 0, & \text{if } \kappa_1\kappa_2\cdots\kappa_s \text{ is not a cotree} \end{cases} \tag{6}$$

In order to give a proof to the theorem on the decision of relative sign graphically, the following four lemmata are needed.

Lemma 1. The value of tree determinant is decided by only one term of all the permutations of indices, which is represented as

$$m!D^{[1}_{\kappa_1} D^{2}_{\kappa_2} \cdots D^{m]}_{\kappa_m} = (-1)^{\alpha} D^{1}_{\lambda_1} D^{2}_{\lambda_2} \cdots D^{m}_{\lambda_m} \tag{7}$$

where $\lambda_1\lambda_2\cdots\lambda_m$ is a certain permutation with respect to $\kappa_1\kappa_2\cdots\kappa_m$ and α is the number of inversion of $\lambda_1\lambda_2\cdots\lambda_m$.

Proof : firstly let it be proved that the branches of a tree correspond in a one-to-one manner to the independent vertices uniquely. This fact is verified by induction on the number of independent vertices. Note that the graph con tain s m independent vertices. If $m=1$, it is evident that the correspondence is unique since tree can contain only one branch. Let the lemma be true for $m = r$. For a tree with $r+1$ independent vertices there exists at least one vertex of degree 1. Otherwise tree contains at least one circuit. Let κ be branch incident to vetex a of degree 1. If κ and a are removed from tree, the result contains r independent vertices. By induction hypothesis, the one-to-one correspondence exists uniquely. Let restore κ and a as it was, and let κ correspond to vertex a, then there is the unique one-to-one correspondence between branches and independent vertices. Hence we get the lemma. (Q.E.D.)

Suppose the graph contains $m+1$ vertices, a tree of G consists of m branches. Tree t_1 and t_2 are said to be adjacent if t_1 and t_2 share $m-1$ branches and differ in only one branch.

Lemma 2. Let tree t_1 and t_2 be adjacent. Branch κ is contained in t_1 but not in t_2, and branch λ is in t_2 but not in t_1. Circuit C which contains both κ and λ comes into being if κ (or λ) is added to t_2 (or t_1). Relative sign for t_1 and t_2 is positive if κ has the same direction as λ in C. Otherwise relative sign is negative.

Proof : If branch κ (or λ) is removed from circuit C, the result is tree t_2 (or t_1). Hence C must contain both κ and λ. Suppose that r vertices are contained in C, then there must be r branches which constitute C. By lemma 1, the one-to-one correspondence exists between the independent vertices and branches of tree, so that $r-1$ vertices in C correspond with $r-1$ branches in C. Any one vertex in C must correspond with the branch which is incident to this vertex but not contained in C. Without loss of generality, we can assume that vertex v_{s+1} is such one, where

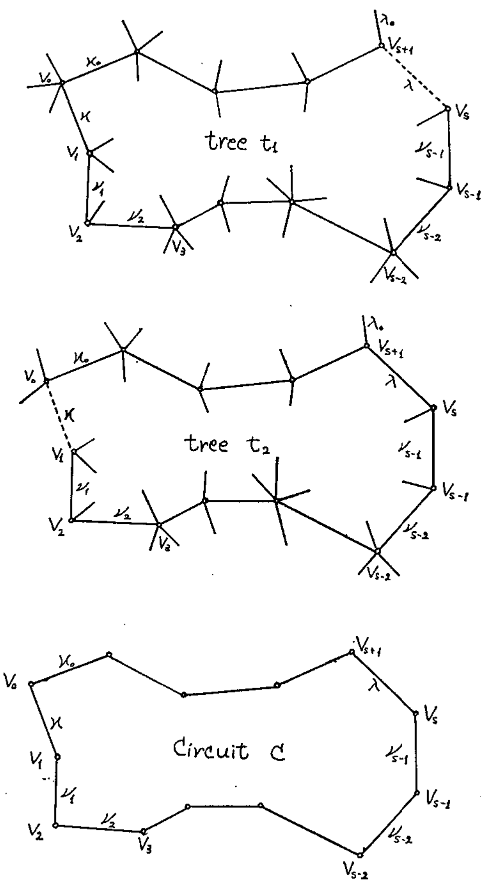


Fig. 1

we number the vertices and branches as in Fig. 1. Suppose that branch κ corresponds with vertex v_0. From (7) determinant of t_1 is given by

$$(-1)^{\alpha} D^{1}_{\lambda_1} \cdots D^{v_0}_{\kappa_0}\, D^{v_1}_{\kappa}\, D^{v_2}_{\nu_1} \cdots D^{v_{s-1}}_{\nu_{s-2}}\, D^{v_s}_{\nu_{s-1}}\, D^{v_{s+1}}_{\lambda_0} \cdots D^{m}_{\lambda_m} \tag{8}$$

where α is the number of inversion of permutation $\lambda_1 \lambda_2 \cdots \kappa_0\, \kappa\nu_1 \cdots \nu_{s-2}\nu_{s-1}\lambda \cdots \lambda_m$ with regard to permutation $\kappa_1\kappa_2\cdots\kappa_m$. Remove κ from t_1 and add λ to the rest, and we get t_2. Then $s-1$ substitutions yield in $\lambda_1\lambda_2\cdots\kappa_0\kappa\nu_1\cdots\nu_{s-2}\nu_{s-1}\lambda\cdots\lambda_m$. Since D^{a}_{κ} in (8) is the element of the determinant, the tree determinant of t_2 can be given by

$$(-1)^{\alpha+s-1} D^{1}_{\lambda_1} \cdots D^{v_0}_{\kappa_0}\, D^{v_1}_{\nu_1}\, D^{v_2}_{\nu_2} \cdots D^{v_{s-1}}_{\nu_{s-1}}\, D^{v_s}_{\lambda}\, D^{v_{s+1}}_{\lambda_0} \cdots D^{m}_{\lambda_m} \tag{9}$$

Relative sign for t_1 and t_2 is given by the product (8)×(9).
Since

$$D^{a}_{\kappa} = \pm 1, \quad \text{and} \quad D^{v_i}_{\nu_i} D^{v_{i+1}}_{\nu_i} = -1 \ (1 \leq i < s-1),$$

the product is

$$(-1)^{2\alpha+s-1} D^{v_1}_{\kappa}(D^{v_1}_{\nu_1} D^{v_2}_{\nu_2})\cdots(D^{v_{s-1}}_{\nu_{s-1}} D^{v_s}_{\nu_{s-1}}) D^{v_s}_{\lambda} \tag{10}$$

$$= D^{v_1}_{\kappa} D^{v_s}_{\lambda}$$

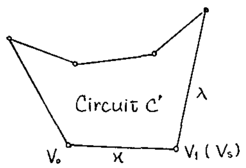


Fig. 2

From (10) it follows that relative sign for t_1 and t_2 has nothing to do with the branches between vertex v_1 and v_s. (See Fig. 1). Therefore we have only to remark the fact that v_1 and v_s are connected, and we can remove branches $\nu_1, \nu_2, \cdots \nu_{s-1}$ by shortening. Consequently we get circuit C′. Henceforth, relative sign can be investigated in C′ instead of C. (See Fig. 2.) If both κ, λ go into or out of vertex v_1, the value of (10) is $+1$. In this case κ, λ have different direction each other in C′. Otherwise (10) is -1 and κ, λ are same in the direction in C′. (Q.E.D.)

Lemma 3. There exists the relationship between relative sign of adjacent trees and that of corresponding cotrees such as

$$\frac{D_{[\kappa_1\cdots\kappa\cdots\kappa_m]}}{D_{[\kappa_1\cdots\lambda\cdots\kappa_m]}} = (-1)\frac{R^{[\nu_1\cdots\lambda\cdots\nu_s]}}{R^{[\nu_1\cdots\kappa\cdots\nu_s]}} \ (1 < \cdots < \kappa < \cdots < \lambda < \cdots < n) \tag{11}$$

where $D_{[\kappa_1\cdots\kappa\cdots\kappa_m]} = D^{[1}_{\kappa_1} D^{2}_{\kappa_2} \cdots D^{m]}_{\kappa_m}$, $R^{[\nu_1\cdots\lambda\cdots\nu_s]} = R^{\nu_1}_{[1} R^{\nu_2}_{2} \cdots R^{\nu_s}_{s]}$.

Proof : Consider the determinant of matrix $\begin{pmatrix} D^{a}_{\kappa} \\ R^{\kappa}_{p} \end{pmatrix}$, By Laplace′s expnsion by first m rows, this is represented as

$$\det\begin{pmatrix} D^{a}_{\kappa} \\ R^{\kappa}_{p} \end{pmatrix} = n!m!\, D^{[12\cdots m]}_{h_1h_2\cdots h_m}\, A^{h_1h_2\cdots h_m}_{[12\cdots m}\, R^{m+1}_{[m+1}\cdots R^{n]}_{n]} \tag{12}$$

Replace column κ by λ in (12), and consider only one term such that $h_1, h_2, \cdots, h_m$ correspond with tree $\kappa_1\cdots\kappa\cdots\kappa_m$.
Then

$$D^{[12\cdots m]}_{\kappa_1\cdots\kappa\cdots\kappa_m}\, A^{\kappa_1\cdots\kappa_m}_{[12\cdots m}\, R^{m+1}_{[m+1}\cdots R^{n]}_{u]}$$

$$= (-1) D^{[12\cdots m]}_{\substack{\kappa_1\cdots\kappa\cdots\kappa_m \\ (\lambda)}}\, A^{\overset{(\lambda)}{\kappa_1\cdots\kappa\cdots\kappa_m}}_{[12\cdots m}\, R^{m+1\cdots n]}_{[m+1\cdots n]} \tag{13}$$

Since we need only sign, (13) can be written as

$$(-1)^{\alpha} D_{[\kappa_1\cdots\kappa\cdots\kappa_m]} R^{[\nu_1\cdots\nu\cdots\nu_s]} = (-1)^{\alpha+1} D_{[\kappa_1\cdots\underset{(\lambda)}{\kappa}\cdots\kappa_m]} R^{[\nu_1\cdots\overset{(\lambda)}{\kappa}\cdots\nu_s]}$$

where $\alpha = \kappa_1+\cdots+\kappa_m$, and (κ) denotes the original numbering. Adopting original numbering again, we get

$$\frac{D_{[\kappa_1\cdots\kappa\cdots\kappa_m]}}{D_{[\kappa_1\cdots\lambda\cdots\kappa_m]}} = (-1)\, \frac{R^{[\nu_1\cdots\lambda\cdots\nu_s]}}{R^{[\nu_1\cdots\kappa\cdots\nu_s]}} \quad \text{(Q.E.D.)}$$

The next lemma on the existence of a Hamilton circuit in a tree graph was due to R. L. Cummins [5] and T. Kamae. [6]

Lemma 4 . There exists a Hamilton circuit in a tree graph.

The tree graph associated with graph G is a graph in which the vertices are in one-to-one correspondence with the trees of G and in which the branches represent the adjacencies of trees. A Hamilton circuit is a circuit which contains all the vertices in a graph. From lemma 4 it is understood that we can get tree t_2 from tree t_1 by substituting the branches one by one. By means of above four lemmata the next theorem on relative sign can be proved completely.

Theorem 1 . Relative sign is decided by a following manner.

(i) Let t_1 and t_2 be adjacent trees which differ in branches κ, λ ($\kappa \in t_1$ $\lambda \in t_2$), and let C be a circuit obtained by adding κ to t_2. Relative sign is positive if κ, λ is different in the direction in C. Otherwise relative sign is negative. If t_1 and t_2 are not adjacent, relative sign is obtained by successive repetition of above process.

(ii) Let t_1 and t_2 be adjacent cotrees differing in branches κ, λ ($\lambda \in t_1$

$\kappa \in t_2$), and let C be a circuit obtained by adding κ to t_1. Relative sign is positive if κ, λ have the same direction in C. Otherwise it is negative. If t_1 and t_2 are not adjacent, relative sign is decided by the successive repetition of this process.

Proof : By lemma 2 and 4, (i) is evident. From (i) and lemm 3, (ii) has been proved. (Q.E.D.)

4. Enumeration of the Number of Several Kinds of Trees and Cotrees

Let G be a given graph, and let G_1, G_2 be obtained by coalescing the independent vetices a, b into dependent vertex v respectively. Such process is denoted by ⓐ + ⓥ or ⓑ + ⓥ. Let D, D_1 and D_2 be incidence matrices of G, G_1 and G_2. By a common tree (or cotree) we mean a tree (or cotree) contained in common both G_1 and G_2.

Theorem 2. The number of common tree is given by

$$T^a_b = (-1)^{a+b}\det(D_1D_2')$$

which is equal to (a, b) co-factor of $\det(D^a_\kappa A^{\kappa\lambda} D^b_\lambda)$.

Proof : By the use of Binet-Cauchy's expansion (1),

$$\det\ (D_1\ D_2')$$

$$= \det(D_1AD_2')$$

$$=(m-1)!D^{[1}_{\kappa_2}\overset{\bar{a}}{\cdots} D^{m]}_{\kappa_m}\ A^{\kappa_2\lambda_2}\cdots A^{\kappa_m\lambda_m}\ D^{[1}_{\lambda_2}\overset{\bar{b}}{\cdots} D^{m]}_{\lambda_m} \tag{14}$$

$$=m!m(-1)^{a+b}A^{[1}_{a}D^2_{\kappa_2}\cdots D^{m]}_{\kappa_m}\ A^{\kappa_2\lambda^2}\cdots A^{\kappa_m\lambda_m}\ A^{[1}_{b}D^2_{\lambda_2}\cdots D^{m]}_{\lambda_m} \tag{15}$$

where indices $\kappa_2, \kappa_3, \cdots \kappa_m, \lambda_2, \lambda_3, \cdots \lambda_m$ range over the integers $1, 2, \cdots, n$ and a, b mean that indices a, b are not in alternation. From (15) it is concluded that there exist terms if and only if $\kappa_i = \lambda_i$ ($i=2, 3, \cdots, m$) and $\kappa_2 \cdots \kappa_s$ corresponds with a sub-tree of G.

The sign of the product

$$D^{[1}_{\kappa_2}\overset{\bar{a}}{\cdots} D^{m]}_{\kappa_m} \times D^{[1}_{\lambda_2}\overset{\bar{b}}{\cdots} D^{m]}_{\lambda_m} \tag{16}$$

can be decided in the following manner.

Let M be the determinant for tree $\kappa_0\kappa_2\cdots\kappa_m$. The (a, κ_0) and (b, κ_0) minors of M are denoted by M_1 and M_2 respectively.

Since the value of M is $+1$ or -1,

$$M_1 = (-1)^{a+\kappa_0} M, \quad M_2 = (-1)^{b+\kappa_0} M.$$

Then

$$M_1 \times M_2 = (-1)^{a+b}$$

which is equal to (16).
Hence the sign of each term in (15) is all same. Whereas by Binet-Cauchy's expansion,

$$\det(D^a_\kappa A^{\kappa\lambda} D^b_\lambda)$$

$$= m!(D^{[1}_{\kappa_1} A^{|\kappa_1\lambda_1|} D^{[1}_{\lambda_1})(D^{]2}_{\kappa_2} A^{|\kappa_2\lambda_2|} D^2_{\lambda_2})\cdots(D^{m]}_{\kappa_m} A^{|\kappa_m\lambda_m|} D^{m]}_{\lambda_m})$$

$$= m! D^{[1}_{\kappa_1} D^2_{\kappa_2} \cdots D^{m]}_{\kappa_m} A^{\kappa_1\lambda_1} A^{\kappa_2\lambda_2} \cdots A^{\kappa_m\lambda_m} D^{[1}_{\lambda_1} D^2_{\lambda_2} \cdots D^{m]}_{\lambda_m} \tag{17}$$

where indices $\kappa_1, \cdots \kappa_m$, $\lambda_1, \cdots, \lambda_m$ range over $1, 2, \cdots, n$.
By (3), (a, b) co-factor of (17) is expressed as

$$m!m\, A^{[1}_a A^{[1}_b (D^2_{\kappa_2} A^{|\kappa_2\lambda_2|} D^2_{\lambda_2})\cdots(D^{m]}_{\kappa_m} A^{|\kappa_m\lambda_m|} D^{m]}_{\lambda_m})$$

$$= m!m\, A^{[1}_a D^2_{\kappa_2} \cdots D^{m]}_{\lambda_m} A^{\kappa_2\lambda_2} \cdots A^{\kappa_m\lambda_m} A^{[1}_b D^2_{\lambda_2} \cdots D^{m]}_{\lambda_m} \tag{18}$$

where $\kappa_2, \cdots, \kappa_m$, $\lambda_2, \cdots \lambda_m$ are the integers from 1 to n.
Compare (18) with (14), we get the result. (Q.E.D.)

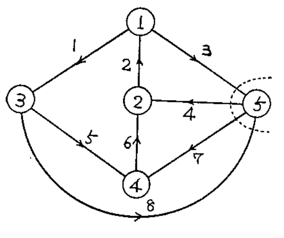


Fig. 3

Hereafter the graph G in Fig. 3 is taken as an example. The dependent vertex is excluded by a dotted arc. All trees and cotees of G are obtained by the method of tree generation presented by M. Piekarski[7] and T. Mizuma.[8] The value of $\det(D^a_\kappa A^{\kappa\lambda} D^b_\lambda)$ for G is given by

$$T = \det(D^a_\kappa A^{\kappa\lambda} R^b_\lambda) = \begin{vmatrix} 3 & -1 & -1 & 0 \\ -1 & 3 & 0 & -1 \\ -1 & 0 & 3 & -1 \\ 0 & -1 & -1 & 3 \end{vmatrix} = 45$$

which is the number of all trees of G.

Example 1. Let G_1 be obtained by process ①+⑤ and G_2 by ②+⑤.

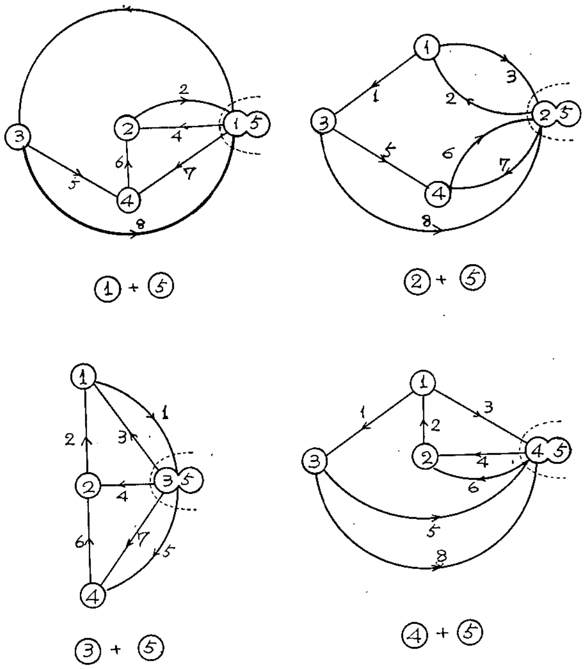


Fig. 4

$$T_2^1 = (-1)^{1+2}\begin{vmatrix} -1 & 0 & -1 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{vmatrix} = 9$$

The common trees calculated are 125, 126, 127, 156, 256, 257, 258, 268, 278.

Example 2. G_1 : ②+⑤, G_2 : ③+⑤

$$T_3^2 = (-1)^{2+3}\begin{vmatrix} 3 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 3 \end{vmatrix} = 6$$

Common trees; 125, 126, 145, 156, 235, 245, 256, 345, 356.

Example 3. G_1 : ③+⑤, G_2 : ④+⑤

$$T_4^3 = (-1)^{3+4}\begin{vmatrix} 3 & -1 & -1 \\ -1 & 3 & 0 \\ 0 & -1 & -1 \end{vmatrix} = 9$$

Common trees : 125, 126, 136, 156, 168, 256, 268, 356, 368

Let G_1 and G_2 be the graphs obtained by process (κ) and (λ), and let R, R_1, and R_2 be the f-circuit matrices of G, G_1, and G_2 respectively.

Theorem 3. The difference between the number of common cotrees with positive relative sign and that with negative relative sign is given by

$$T^{p}_{q} = \left| (p,q) \text{ co-factor of } \det(D^{\kappa}_{p} A_{\kappa\lambda} R^{\lambda}_{p}) \right|$$

where $| \quad |$ means the absolute value.

Proof :

By the use of (1)

$$T = \det(R^{\kappa}_{p} A_{\lambda\kappa} R^{\lambda}_{q})$$

$$= s! R^{\kappa_1}_{[1} R^{\kappa_2}_{2} \cdots R^{\kappa_s}_{s]} A_{\kappa_1\lambda_1} \cdots A_{\kappa_s\lambda_s} R^{\lambda_1}_{[1} R^{\lambda_2}_{2} \cdots R^{\lambda_s}_{s]}$$

where indices $\kappa_1 \cdots \kappa_s$, $\lambda_1 \cdots \lambda_s$ take the values $1, 2, \cdots, n$.

By (1) again, (p, q) co-factor is written as

$$T^{p}_{q} = s! s A^{p}_{[1} R^{\kappa_2}_{2} \cdots R^{\kappa_s}_{s]} A_{\kappa_2\lambda_2} \cdots A_{\kappa_s\lambda_s} A^{q}_{[1} R^{\lambda_2}_{2} \cdots R^{\lambda_s}_{s]} \tag{19}$$

If circuit matrix of G is taken as f-circuit one, (19) is expressed as

$$s! s R^{\kappa}_{[1} R^{\kappa_2}_{2} \cdots R^{\kappa_s}_{s]} A_{\kappa_2\lambda_2} \cdots A_{\kappa_s\lambda_s} R^{\lambda}_{[1} R^{\lambda_2}_{2} \cdots R^{\lambda_s}_{s]} \tag{19$'$}$$

where branches κ, λ constitute circuits p, q respectively. In (19) there exist only terms such that $\kappa_i = \lambda_i$ $(i = 2, 3, \cdots, s,\ \kappa_i \neq \kappa, \lambda)$ and $\kappa_2 \kappa_3 \cdots \kappa_s$ corresponds with a sub-cotree of G. Considering relative sign of each sub-cotree in (19)′, we get the theorem. (Q.E.D.)

Let G $(p \backslash \kappa)$ and G $(q \backslash \lambda)$ be obtained by the processes $(\bar{\kappa})$ and $(\bar{\lambda})$ respecttively. The system of f-circuits is obtained by adding branches 2345 to tree 1678. It is the set of circuits 138, 467, 578, 12678.

Example 4. $G_1 = G(1\backslash 2)$, $G = G_2(2\backslash 3)$. See Fig. 5.

Since the number of all cotree of G is geivn by

$$T = \begin{vmatrix} 5 & -2 & -2 & -2 \\ -2 & 3 & 0 & 1 \\ -2 & 0 & 3 & 1 \\ -2 & 1 & 1 & 3 \end{vmatrix} = 45$$

then

$$T^{1}_{2} = \left| \det \begin{pmatrix} -2 & 0 & 1 \\ -2 & 3 & 1 \\ -2 & 1 & 3 \end{pmatrix} \right| = 12$$

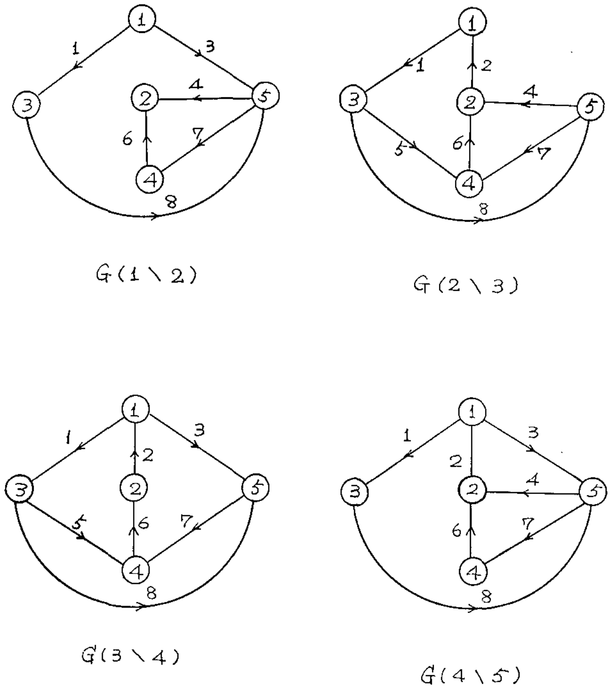


Fig. 5

f- Circuits

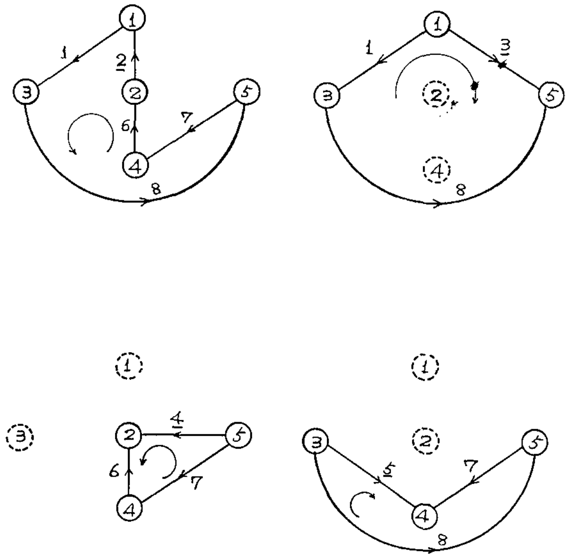


Fig. 6

which is the number of common cotrees in G_1 and G_2 since all the common cotrees have the same sign in this case.
common cotrees : 145, 147, 148, 156, 157, 167, 168, 178, 458, 568, 678.

Example 5 . $G_1 = G(2\backslash 3)$, $G_2 = G(3\backslash 4)$.

$$T_3^2 = \left| \det \begin{pmatrix} 5 & -2 & -2 \\ -2 & 0 & 1 \\ -2 & 1 & 3 \end{pmatrix} \right| = 9$$

which gives all common cotrees in the same reason as above example.
Common cotrees : 156, 157, 167, 168, 178, 278, 568, 578, 678.

Example 9 . $G_1 = G_2(3/4)$, $G = G(4/5)$

$$T_4^3 = \left| \det \begin{pmatrix} 5 & -2 & -2 \\ -2 & 3 & 0 \\ -2 & 1 & 1 \end{pmatrix} \right| = 3$$

Common cotrees with poitive relative sign is 368 and those with negative sign are 127, 137, 237, 278.

Let G_1 and G_2 be the graphs obtained from G by process $(\bar{\kappa}\ \lambda)$ and $(\kappa\ \bar{\lambda})$ respectively. The adjacent trees which differ in branches κ, λ are considered. Let D_1, D_2 be the incidence matrices, and R_1, R_2 be the circuit matrices of G_1 and G_2.

Eheoem 4 . The difference between the number of adjacent trees with positive relative sign and that with negative relative sign is given by

$${}_\kappa T_\lambda = |\det(D_1 D_2')|$$

or

$${}_\kappa T_\lambda = |\det(R_1 R_2')|$$

Proof :

$$\det(D_1 D_2')$$
$$= \det(D_1 A\ D_2')$$

$$=(m-1)!D_{\kappa_2}^{[1\dot{a}} \cdots D_{\kappa_m}^{m]} A^{\kappa_2\lambda_2} \cdots A^{\kappa_m\lambda_m} D_{\lambda_2}^{[1\dot{b}} \cdots D_{\lambda_m}^{m]} \tag{20}$$

where branch κ, λ are not contained both in tree $\kappa_2 \cdots \kappa_m$, of G and tree $\lambda_2 \cdots \lambda_m$ of G, and a means that vertex a is excluded by shortening. By lemma 1, (20) can be represented as

$$m!m\alpha D_{\kappa}^{[1} D_{\kappa_2}^{2} \cdots D_{\kappa_m}^{m]} A^{\kappa_2\lambda_2} \cdots A^{\kappa_m\lambda_m} D_{\lambda}^{[1} D_{\lambda_2}^{2} \cdots D_{\lambda_m}^{m]} \tag{21}$$

where $\alpha = D_\kappa^a D_\lambda^b$.

There exist terms in (21) such that $\kappa_i = \lambda_i$ ($i = 2, 3, \cdots, m$), and and $\kappa\ \kappa_2 \cdots \kappa_m$ corresponds with a tree. Relative sign for tree $\kappa\kappa_2\cdots\kappa_m$ and $\lambda\kappa_2 \cdots \kappa_m$ is decided by theorem 1. Therefore ${}_\kappa T_\lambda$ gives the difference between the number of adjacent tree with positive sign and that with negative sign.

Whereas it is assumed that R, R_1, R_2 are taken as fundamental circuit matrices.

Then,

$$\det(R_1R_2')$$

$$= \det(R_1AR_2')$$

$$=(s-1)!R^{\kappa_2}_{[1}\cdots_{\underline{q}}R^{\kappa_s}_{s]} A_{\kappa_2\lambda_2}\cdots A_{\kappa_m\lambda_m} R^{\lambda_2}_{[1}\cdots_{\underline{q}}R^{\lambda_s}_{s]} \tag{22}$$

where branch κ is contained in only circuit p, and λ is in only q. Since R is taken as f-circuit matrices, (22) can be written as

$$s!s\ \beta R^{\kappa}_{[1} R^{\kappa_2}_{2}\cdots R^{\kappa_s}_{s]} A_{\kappa_2\lambda_2}\cdots A_{\kappa_m\lambda_m} R^{\lambda}_{[1}{}^{\lambda_2}_{2}\cdots R^{\lambda_s}_{s]} \tag{23}$$

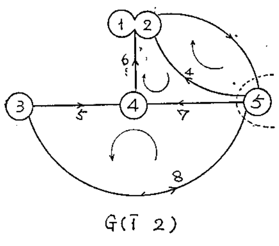
where $\beta = R^{\kappa}_{p} R^{\lambda}_{q}$ and $\kappa_2,\cdots\kappa_s,\lambda_2,\cdots\lambda_s$ do not contain both κ. anb

By (23) there remain only the terms such that $\kappa_i = \lambda_i$ ($i=2, 3, \cdots, s$) and $\kappa\kappa_2\cdots\kappa_s$ and $\lambda\lambda_2\cdots\lambda_s$ correspond with a tree. Relative sign is determined according to theorem 1. Hence (23) gives the difference of the number of the adjacent cotrees with positive sign and that with negative sign. Since there is the one-to-one correspondence between the trees and cotrees of the graph,

${}_\kappa T_\lambda = {}_\kappa\overline{T}_\lambda$ (Q.E.D.)

The graph obtained by process $(\kappa\ \bar{\lambda})$ is denoted by G $(\kappa\ \bar{\lambda})$, and that by $(\bar{\kappa}\ \lambda)$ is done by G $(\bar{\kappa}\ \lambda)$.

Example 7. $G_1 = G\ (\bar{1}\ 2)$, $G_2 = G\ (1\ \bar{2})$. See Eig. 7 .The f-circuits are taken as in Eig. 6.



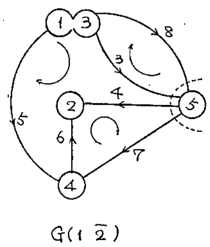


Fig. 7

$$
{}_1T_2 = \left| \det \begin{pmatrix} 1 & -1 & -0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1- & 1 & 0 \end{pmatrix}' \right|
$$

$$
= \left| \det \begin{pmatrix} 1 & 2 & -1 \\ 2 & 0 & -1 \\ -1 & -1 & 3 \end{pmatrix} \right| = 9
$$

$$
{}_1T_3 = \left| \det \begin{pmatrix} 0 & 0 & -1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & -1 & 1- & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \end{pmatrix}' \right|
$$

$$
= \left| \det \begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \\ 0 & -2 & 3 \end{pmatrix} \right| = 9
$$

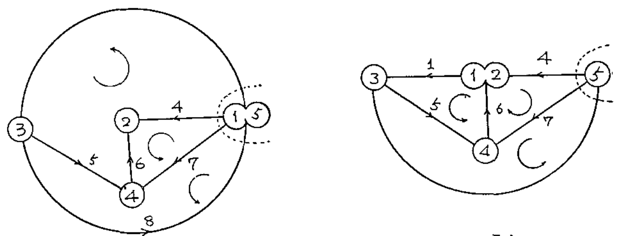
which give the number of all adjcent cotrees differing in branch 1, 2 since all relative sign are same.

Adjacent cotrees :

$\begin{bmatrix}1345\\2345\end{bmatrix}$ $\begin{bmatrix}1347\\2347\end{bmatrix}$ $\begin{bmatrix}1348\\2348\end{bmatrix}$ $\begin{bmatrix}1356\\2356\end{bmatrix}$ $\begin{bmatrix}1357\\2357\end{bmatrix}$ $\begin{bmatrix}1367\\2367\end{bmatrix}$ $\begin{bmatrix}1368\\2368\end{bmatrix}$ $\begin{bmatrix}1378\\2378\end{bmatrix}$ $\begin{bmatrix}1478\\2478\end{bmatrix}$

The adjacent trees are easily obtained from this result.

Example 8 . $G_1 = G(\bar{2}\ \ 3)$, $G_2 = G(2\ \ \bar{3})$. See Fig. 8 .



G($\bar{2}$ 3)　　G(2 $\bar{3}$)

Fig. 8

$$
{}_2T_3 = \left| \det \begin{pmatrix} 0 & -1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -1 & 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 1- & 0 \end{pmatrix}' \right|
$$

$$= \left| \det \begin{pmatrix} 2 & 0 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix} \right| = 12$$

$${}_2T_3 = \left| \det \begin{pmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 1 \end{pmatrix}' \right|$$

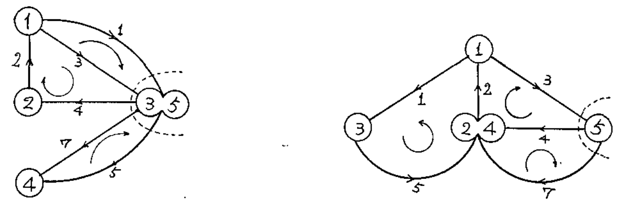
$$= \left| \det \begin{pmatrix} 3 & 2 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix} \right| = 12$$

which give the number of all adjacent cotrees.

Adjacent cotrees : $\left[\begin{smallmatrix}2145\\3145\end{smallmatrix}\right.$ $\left[\begin{smallmatrix}2147\\3147\end{smallmatrix}\right.$ $\left[\begin{smallmatrix}2148\\3148\end{smallmatrix}\right.$ $\left[\begin{smallmatrix}2156\\3156\end{smallmatrix}\right.$ $\left[\begin{smallmatrix}2167\\3167\end{smallmatrix}\right.$ $\left[\begin{smallmatrix}2168\\3168\end{smallmatrix}\right.$ $\left[\begin{smallmatrix}2178\\3178\end{smallmatrix}\right.$ $\left[\begin{smallmatrix}2458\\3458\end{smallmatrix}\right.$

$\left[\begin{smallmatrix}2568\\3568\end{smallmatrix}\right.$ $\left[\begin{smallmatrix}2578\\3578\end{smallmatrix}\right.$ $\left[\begin{smallmatrix}2678\\3678\end{smallmatrix}\right.$

Example 9. $G = G(\overline{6}\ 8)$, $G = G(6\ \overline{8})$. See Fig. 9.



$G(\overline{6}\ 8)$　　　　$G(6\ \overline{8})$

Fig. 9

$${}_6T_8 = \det \left| \begin{pmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & -1 \\ -1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}' \right|$$

$$= \det \left| \begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & 0 \\ 0 & 2 & -1 \end{pmatrix} \right| = 3$$

$$
{}_6T_8 = \left| \det \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}' \right|
$$

$$
= \left| \det \begin{pmatrix} 1 & -1 & 0 \\ 1 & 3 & -1 \\ 1 & 0 & -1 \end{pmatrix} \right| = 3
$$

Ahjacent cotrees with positive relativ esign : $\left[\begin{smallmatrix}6345\\8345\end{smallmatrix}\right.$

Ahjacent cotrees with negative relative sign : $\left[\begin{smallmatrix}6237\\8237\end{smallmatrix}\right. \left[\begin{smallmatrix}6147\\8147\end{smallmatrix}\right. \left[\begin{smallmatrix}6127\\8127\end{smallmatrix}\right. \left[\begin{smallmatrix}6137\\8137\end{smallmatrix}\right.$

5. Applications to Topological Analysis of Electrical Networks

The passive network consisting of n branches, m+1 nodes, and s independent circuits considered. In this case node means vertex. Suppose that impedance $z_{\kappa\lambda}$ and voltage source e_λ are in each branch. For the sake of brevity, it is assumed that no mutual impedances exist.

Theorem 5. Getting current functions is equivalent to obtain

1) all cotrees
2) adjacent cotrees
3) relative signs between adjacent cotrees

of the graph associated with a given network diagram.

Proof : By Kirchhoff's voltage law,

$$R_q^\lambda u\lambda = R_q^\lambda \ e_\lambda$$

By Ohm's law,

$$u_\lambda = z_{\lambda\kappa} \ i^\kappa$$

From the relation between branch and circuit current,

$$i^\kappa = R_p^\kappa \ J^p$$

Combination of above three eluations leads to

$$i^\kappa = R_p^\kappa (R_q^\lambda \ z_{\lambda\kappa} \ R_p^\kappa)^{-1} R_q^\lambda e_\lambda \tag{24}$$

Let

$$y^{pq} = (R_q^\lambda \ z_{\lambda\kappa} \ R_p^\kappa)^{-1} \tag{25}$$

and let

$$R^{\kappa\lambda} = R_p^{\kappa}\, y^{pq}\, R_q^{\lambda} \tag{26}$$

Then i^{κ} is expressed as

$$i^{\kappa} = R^{\kappa\lambda}\, e_{\lambda}$$

By means of Bint-Cauchy's expansion (1) and co-factor (3),

$$y^{pq} = \frac{s!s\, A^{p}_{[1}\, R^{\kappa_2}_{2} \cdots R^{\kappa_s}_{s]}\, z_{\kappa_2\lambda_2} \cdots z_{\kappa_s\lambda_s}\, A^{q}_{1]}\, R^{\lambda_2}_{2} \cdots R^{\lambda_s}_{s]}}{s!\, R^{\kappa_1}_{[1}\, R^{\kappa_2}_{2} \cdots R^{\kappa_s}_{s]}\, z_{\kappa_1\lambda_1}\, z_{\kappa_2\lambda_2} \cdots z_{\kappa_s\lambda_s}\, R^{\lambda_1}_{[1}\, R^{\lambda_3}_{2} \cdots R^{\lambda_s}_{s]}} \tag{27}$$

Let

$$Z = s!R^{\kappa_1}_{[1}\, R^{\kappa_2}_{2} \cdots R^{\kappa_s}_{s]}\, z_{\kappa_1\lambda_1}\, z_{\kappa_2\lambda_3} \cdots z_{\kappa_s\lambda_s}\, R^{\lambda_1}_{[1}\, R^{\lambda_2}_{2} \cdots R^{\lambda_s}_{s]} \tag{28}$$

$$Z^{pq} = s!s\, A^{p}_{[1}\, R^{\kappa_2}_{2} \cdots R^{\kappa_s}_{s]}\, z_{\kappa_2\lambda_2}\, z_{\kappa_3\lambda_3} \cdots z_{\kappa_s\lambda_s}\, A^{q}_{[1}\, R^{\lambda_2}_{2} \cdots R^{\lambda_s}_{s]}$$

Using (31) in (25), we get

$$R^{\kappa\lambda} = s!s\, R^{\kappa}_{p}\, A^{p}_{[1}\, R^{\kappa_2}_{2} \cdots R^{\kappa_s}_{s]}\, z_{\kappa_2\lambda_2} \cdots z_{\kappa_s\lambda_s}\, R^{\lambda}_{q}\, A^{q}_{[1}\, R^{\lambda_2}_{2} \cdots R^{\lambda_s}_{s]}/Z \tag{29}$$

where branches κ, λ constitute circuits p, q respectively.
Let

$$Z^{\kappa\lambda} = s!s\, R^{\kappa}_{[1}\, R^{\kappa_2}_{2} \cdots R^{\kappa_s}_{s]}\, z_{\kappa_2\lambda_2} \cdots z_{\kappa_s\lambda_s}\, R^{\lambda}_{[1}\, R^{\lambda_2}_{2} \cdots R^{\lambda_s}_{s]} \tag{31}$$

Then

$$R^{\kappa\lambda} = Z^{\kappa\lambda}/Z \tag{32}$$

It is obvious that from (27), (30) to obtain Z is equal to get all cotrees of a graph and to obtain $Z^{\kappa\lambda}$ is to get adjacent trees differing in κ, λ and their relative signs. Considering the proof of theorem 3, we get the theorem. (Q.E.D.)

Example 10. Network shown in Eig.10 is taken as an example. In this case network consists of 5nodes, 8 branches and (8 − 5 + 1) = 4 independent-circuits.

Then,

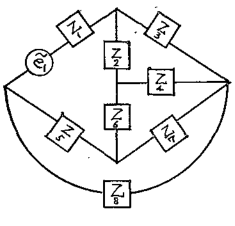


Fig. 10

$$i^{\kappa} = R^{\kappa 1} e_1 ,$$

where index κ ranges from 1 to 8.

$$Z = 4!R^{\kappa_1}_{[1} R^{\kappa_2}_{2} R^{\kappa_3}_{3} R^{\kappa_4}_{4]} z_{\kappa_1\lambda_1} z_{\kappa_2\lambda_2} z_{\kappa_3\lambda_3} z_{\kappa_4\lambda_4} R^{\lambda_1}_{[1} R^{\lambda_2}_{2} R^{\lambda_3}_{3} R^{\lambda_4}_{4]}$$

$$Z^{\kappa 1} = 4!\,4\,R^{\kappa}_{[1} R^{\kappa_2}_{2} R^{\kappa_3}_{3} R^{\kappa_4}_{4]} z_{\kappa_2\lambda_2} z_{\kappa_3\lambda_3} z_{\kappa_4\lambda_4} R^{1}_{[1} R^{\lambda_2}_{2} R^{\lambda_3}_{3} R^{\lambda_4}_{4]}$$

Then,

$$Z = \underset{1245}{zzzz} + \underset{1247}{zzzz} + \underset{1248}{zzzz} + \underset{1256}{zzzz} + \underset{1257}{zzzz} + \underset{1267}{zzzz} + \underset{1268}{zzzz} + \underset{1278}{zzzz} + \underset{1345}{zzzz}$$

$$+ \underset{1347}{zzzz} + \underset{1348}{zzzz} + \underset{\substack{135\\9}}{zzzz} + \underset{1357}{zzzz} + \underset{1367}{zzzz} + \underset{1368}{zzzz} + \underset{1378}{zzzz} + \underset{1456}{zzzz} + \underset{1457}{zzzz}$$

$$+ \underset{1467}{zzzz} + \underset{1468}{zzzz} + \underset{1478}{zzzz} + \underset{2346}{zzzz} + \underset{2347}{zzzz} + \underset{2348}{zzzz} + \underset{2356}{zzzz} + \underset{2357}{zzzz} + \underset{2367}{zzzz}$$

$$+ \underset{2368}{zzzz} + \underset{2378}{zzzz} + \underset{2458}{zzzz} + \underset{2478}{zzzz} + \underset{2563}{zzzz} + \underset{2578}{zzzz} + \underset{2678}{zzzz} + \underset{3456}{zzzz} + \underset{3457}{zzzz}$$

$$+ \underset{3458}{zzzz} + \underset{3467}{zzzz} + \underset{3468}{zzzz} + \underset{3568}{zzzz} + \underset{3578}{zzzz} + \underset{3678}{zzzz} + \underset{4568}{zzzz} + \underset{4578}{zzzz} + \underset{4678}{zzzz}$$

See the value of T in Chapter 4.

Symbol $\underset{\kappa}{z}$ is used in place of $z_{\kappa\kappa}$ for the sake of brevity.

$$i^1 = (Z^{11}/Z)\, e_1$$

$$Z^{11} = \underset{245}{zzz} + \underset{247}{zzz} + \underset{248}{zzz} + \underset{256}{zzz} + \underset{257}{zzz} + \underset{267}{zzz} + \underset{268}{zzz} + \underset{278}{zzz} + \underset{345}{zzz} + \underset{347}{zzz}$$

$$+ \underset{348}{zzz} + \underset{356}{zzz} + \underset{357}{zzz} + \underset{367}{zzz} + \underset{368}{zzz} + \underset{378}{zzz} + \underset{456}{zzz} + \underset{457}{zzz} + \underset{467}{zzz} + \underset{468}{zzz}$$

$$+ \underset{478}{zzz}$$

$$i^2 = (Z^{21}/Z)\, e_1$$

$$Z^{21} = \underset{345}{zzz} + \underset{347}{zzz} + \underset{348}{zzz} + \underset{356}{zzz} + \underset{357}{zzz} + \underset{367}{zzz} + \underset{368}{zzz} + \underset{378}{zzz} + \underset{478}{zzz}$$

See example 7.

$$i^3 = (Z^{31}/Z)\, e_1$$

$$Z^{31} = -(\underset{245}{zzz} + \underset{247}{zzz} + \underset{248}{zzz} + \underset{256}{zzz} + \underset{257}{zzz} + \underset{267}{zzz} + \underset{268}{zzz} + \underset{278}{zzz} + \underset{456}{zzz} + \underset{457}{zzz}$$

$$+ \underset{467}{zzz} + \underset{468}{zzz})$$

$$i^4 = (Z^{41}/Z) e_1$$

$$Z^{41} = \underset{357}{zzz} + \underset{367}{zzz} + \underset{368}{zzz} + \underset{356}{zzz} - \underset{278}{zzz}$$

$$i^5 = (Z^{51}/Z)\ e_1$$

$$Z^{51} = \underset{248}{zzz} + \underset{268}{zzz} + \underset{278}{zzz} + \underset{347}{zzz} + \underset{348}{zzz} + \underset{368}{zzz} + \underset{378}{zzz} + \underset{468}{zzz} + \underset{478}{zzz}$$

$$i^6 = (Z^{61}/Z)\ e_1$$

$$Z^{61} = \underset{278}{zzz} + \underset{345}{zzz} + \underset{347}{zzz} + \underset{348}{zzz} + \underset{378}{zzz} + \underset{478}{zzz}$$

$$Z^{71} = \underset{345}{zzz} - \underset{248}{zzz} - \underset{268}{zzz} - \underset{368}{zzz} - \underset{468}{zzz}$$

$$Z^{81} = \underset{245}{zzz} + \underset{247}{zzz} + \underset{256}{zzz} + \underset{257}{zzz} + \underset{269}{zzz} + \underset{345}{zzz} + \underset{356}{zzz} + \underset{357}{zzz} + \underset{367}{zzz} + \underset{456}{zzz}$$

$$+ \underset{457}{zzz} + \underset{467}{zzz}$$

Suppose that network contains admittance in each branch, and node current source is in each node.

Theorem 6. Getting voltage functions is equivalent to get

1) all trees
2) adjacent trees
3) relative signs between adacent trees

of the graph corresponding with a network.

Proof: From the relation between node current source and equivalent branch currentsoure,

$$D^a_\kappa S^\kappa = S^a$$

where S^a is node current source.

By Kirchhoff's current law,

$$D^a_\kappa i^\kappa = D^a_\lambda S^\kappa$$

By Ohm's law

$$i^\kappa = y^{\kappa\lambda} u_\lambda$$

From relation between branch voltage drop and point potential,

$$u_\lambda = D^b_\lambda V_b$$

Hence we have

$$u_\lambda = D^b_\lambda (D^a_\kappa y^{\kappa\lambda} D^b_1)^{-1} D^a_\kappa S^\kappa \tag{33}$$

Let

$$Z_{ba} = (D^a_\kappa y^{\kappa\lambda} D^b_\lambda)^{-1} \tag{34}$$

and

$$D_{\lambda\kappa} = D^b_\lambda Z_{ba} D^a_\kappa \tag{35}$$

Using (34), (35) in (33), we have

$$u_\lambda = D_{\lambda\kappa} S^\kappa$$

By Binet-Cauchy's expansion (1) and co-factor(3)

$$Z_{ba} = \frac{m!m A^{[1}_b D^2_{\kappa_2} \cdots D^{m]}_{\kappa_m} y^{\kappa_2\lambda_2} \cdots y^{\kappa_m\lambda_m} A^{[1}_a D^2_{\lambda_2} \cdots D^{m]}_{\lambda_m}}{m! D^{[1}_{\kappa_1} D^2_{\kappa_2} \cdots D^{m]}_{\kappa_m} y^{\kappa_1\lambda_1} \cdots y^{\kappa_m\lambda_m} D^{[1}_{\lambda_1} D \quad \cdots D^{m]}_{\lambda_m}} \tag{36}$$

Using (35) in (34), we get

$$D_{\lambda\kappa} = m!m D^b_\lambda A^{[1}_b D^2_{\kappa_2} \cdots D^{m]}_{\kappa_m} y^{\kappa_2\lambda_2} \cdots y^{\kappa_m\lambda_m} D^a_\kappa A^{[1}_a D^2_{\lambda_2} \cdots D^{m]}_{\lambda_m} / Y \tag{37}$$

where

$$Y = m!\, D^{[1}_{\kappa_1} D^2_{\kappa_2} \cdots D^{m]}_{\kappa_m} y^{\kappa_1\lambda_1} \cdots y^{\kappa_m\lambda_m} D^{[1}_{\lambda_1} D^2_{\lambda_2} \cdots D^{m]}_{\lambda_m} \tag{38}$$

Considering lemma 1, (31) can be written as

$$D_{\lambda\kappa} = m!m\, D^{[1}_\lambda D^2_{\kappa_2} \cdots D^{m]}_{\kappa_m} y^{\kappa_2\lambda_2} \cdots y^{\kappa_m\lambda_m} D^{[1}_\kappa D^2_{\lambda_2} \cdots D^{m]}_{\lambda_m} / Y \tag{39}$$

Let

$$Y_{\lambda\kappa} = m!m\, D^{[1}_\lambda D^2_{\kappa_2} \cdots D^{m]}_{\kappa_m} y^{\kappa_2\lambda_2} \cdots y^{\kappa_m\lambda_m} D^{[1}_\kappa D^2_{\lambda_2} \cdots D^{m]}_{\lambda_m} / Y$$

Then

$$u_\lambda = (Y_{\lambda\kappa} / Y) S^\kappa \tag{40}$$

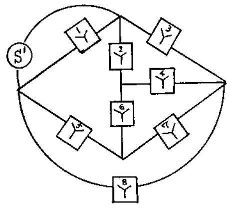


Fig. 11

From (37) and (38) it is evident tha to get Y is epual to obtain all trees of the graph, and to get $Y_{\lambda\kappa}$ is equal to get adjacent trees differing in κ, λ and relative signs. (Q.E.D.)

Example 11. Network diagram shown in Fig.11 is taken. In this case $m=4$, $s=4$. Suppose that

S^1 exists only. Therefore we get

$$Y=4!D^{[1}_{\kappa_1}D^{2}_{\kappa_2}D^{3}_{\kappa_3}D^{4]}_{\kappa_4}y^{\kappa_1\lambda_1}y^{\kappa_2\lambda_2}y^{\kappa_3\lambda_3}y^{\kappa_4\lambda_4}D^{[1}_{\lambda_1}D^{2}_{\lambda_2}D^{3}_{\lambda_3}D^{4]}_{\lambda_4}$$

$$Y_{\lambda_1}=4!4D^{[1}_{\lambda}D^{2}_{\kappa_2}D^{3}_{\kappa_3}D^{4]}_{\kappa_4}y^{\kappa_2\lambda_2}y^{\kappa_3\lambda_3}y^{\kappa_4\lambda_4}D^{[1}_{1}D^{2}_{\lambda_2}D^{3}_{\lambda_3}D^{4]}_{\lambda_4}$$

Hence

$$u_\lambda=(Y_{\lambda 1}/Y)\,S^1$$

$$\begin{aligned}Y=\ &\overset{1235}{yyyy}+\overset{1236}{yyyy}+\overset{1237}{yyyy}+\overset{1245}{yyyy}+\overset{1246}{yyyy}+\overset{1247}{yyyy}+\overset{1257}{yyyy}+\overset{1258}{yyyy}+\overset{1267}{yyyy}\\&+\overset{1268}{yyyy}+\overset{1278}{yyyy}+\overset{1345}{yyyy}+\overset{1346}{yyyy}+\overset{1347}{yyyy}+\overset{1356}{yyyy}+\overset{1367}{yyyy}+\overset{1456}{yyyy}+\overset{1457}{yyyy}\\&+\overset{1458}{yyyy}+\overset{1468}{yyyy}+\overset{1478}{yyyy}+\overset{1567}{yyyy}+\overset{1568}{yyyy}+\overset{1678}{yyyy}+\overset{2356}{yyyy}+\overset{2357}{yyyy}+\overset{2358}{yyyy}\\&+\overset{2368}{yyyy}+\overset{2378}{yyyy}+\overset{2456}{yyyy}+\overset{2457}{yyyy}+\overset{2458}{yyyy}+\overset{2468}{yyyy}+\overset{2478}{yyyy}+\overset{2567}{yyyy}+\overset{2568}{yyyy}\\&+\overset{2678}{yyyy}+\overset{3456}{yyyy}+\overset{3457}{yyyy}+\overset{3458}{yyyy}+\overset{3468}{yyyy}+\overset{3478}{yyyy}+\overset{3567}{yyyy}+\overset{3568}{yyyy}+\overset{3678}{yyyy}\end{aligned}$$

where $\overset{\kappa}{y}$ means $\overset{\kappa\kappa}{y}$.

$$u_1=Y_{11}\,S^1$$

$$\begin{aligned}Y_{11}=\ &\overset{235}{yyy}+\overset{236}{yyy}+\overset{237}{yyy}+\overset{245}{yyy}+\overset{246}{yyy}+\overset{247}{yyy}+\overset{257}{yyy}+\overset{258}{yyy}+\overset{267}{yyy}\\&+\overset{268}{yyy}+\overset{278}{yyy}+\overset{345}{yyy}+\overset{346}{yyy}+\overset{347}{yyy}+\overset{356}{yyy}+\overset{367}{yyy}+\overset{456}{yyy}+\overset{457}{yyy}\\&+\overset{458}{yyy}+\overset{468}{yyy}+\overset{478}{yyy}+\overset{567}{yyy}+\overset{568}{yyy}+\overset{678}{yyy}\end{aligned}$$

$$u_2=(Y_{21}/Y)\,S^1$$

$$Y_{21}=-(\overset{356}{yyy}+\overset{456}{yyy}+\overset{457}{yyy}+\overset{458}{yyy}+\overset{468}{yyy}+\overset{478}{yyy}+\overset{567}{yyy}+\overset{568}{yyy}+\overset{678}{yyy})$$

See example 7.

$$u_3=(Y_{31}/Y)\,S^1$$

$$\begin{aligned}Y_{31}=\ &\overset{257}{yyy}+\overset{258}{yyy}+\overset{267}{yyy}+\overset{268}{yyy}+\overset{278}{yyy}+\overset{456}{yyy}+\overset{457}{yyy}+\overset{458}{yyy}+\overset{468}{yyy}\\&+\overset{478}{yyy}+\overset{567}{yyy}+\overset{586}{yyy}\end{aligned}$$

$$Y_{41}=\overset{356}{yyy}-\overset{257}{yyy}-\overset{258}{yyy}-\overset{267}{yyy}-\overset{278}{yyy}$$

In a similar way we can get u_5, u_6, u_7, u_8.

6. Concluding Remarks

It shoud be noted that lemma 1 given in chapter 3 is useful for another problems which relate to tree determinant.

By the use of lemm 2, specially by Equ. (10) relative sign seems to be calculated by a digital comouter. Theorem 5, 6 is valid if mutual impedances exist. They are also valid if voltage or current sources are in some branches. In such cases analysis can also be carried out by principle of superposition.

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電気回路の位相幾何的解析における二三の定理

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Graph 理論においては，種々の tree あるいは cotree の数を計算すること，および相対符号を計算するこにが重要である。本論文では相対符号次定に関する一定理と，tree, cotree の数を求める方法に関する定理を与え，これらの応用として電気回路解析における基礎的な定理を二つ導く。